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ON AN APPROACH TO THE INVESTIGATION OF THE SIGNORINI PROBLEM USING THE IDEA OF DUALITY*

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Reciprocal variational problems for the boundary functionals of linear elasticity theory, defined on convex closed sets of functions, are formulated in an example of the Signorini problem. Certain unilateral boundary value problems of linear elasticity theory result in variational problems for such functionals. The reciprocity relationship is proved, and error estimates are presented of the approximate solutions of unilateral boundary value problems which can be used, for instance, in solving contact problems of linear elasticity theory.

1. Variational inequalities obtained because of minimizing the functional of a problem in closed convex sets correspond to boundary value problems with unilateral constraints which are described in terms of the desired solution belonging to these sets in a certain Hilbert space /1—4/. The notation in /5—7/ is hereafter retained.

Let $G \subset E_3$ be a bounded domain occupied by an anisotropic elastic medium with a sufficiently smooth boundary $S = S_1 \cup S_2$. The following unilateral problem /1/

$$2\int_G W(u_0, v-u_0)\,dG \geqslant \int_G K(v-u_0)\,dG + \int_{S_2} f(v-u_0)\,ds, \quad u_0 \in V, \quad \forall v \in V \subset W_2^1(G) \tag{1.1}$$

$$W(u,v) = \frac{1}{2}\sum_{i,k,l,m=1}^{3} c_{iklm}(x)\,\varepsilon_{lm}(u)\,\varepsilon_{ik}(v), \quad K \in L_2(G), \quad f \in L_2(S)$$

is understood to be the (generalized) Signorini problem.

Here $K = K(x)$, $x \in G$ is the volume force vector, $f = f(x)$ is a given surface stress vector on S_2, $V \subset W_2^1(G)$ is a closed convex set /1/, which is here made specific in the following manner (which corresponds to the Signorini problem):

$$V = \{v \in W_2^1(G) \mid v^{(\nu)}|_{S_1} \geqslant 0\}$$

where $\nu(x)$ is the unit vector of the internal normal to S (we shall later omit the superscript ν on the vector v).

The problem for the variational inequality (1.1) is equivalent /1/ to the problem of minimizing the functional

$$J(v) = 2\int_G W(v)\,dG - \Psi(v)$$

$$\Psi(v) = 2\int_G Kv\,dG + 2\int_{S_2} fv\,ds$$

on V. Under certain conditions mentioned below that are imposed on $\Psi(v)$ /1/ this problem is solvable and the minimizing function satisfies the vector equilibrium equations $Au_0 = K$ in the domain G, the so-called questionable boundary conditions on S_1 /1/ (in the absence of friction on the surface), $u_0^{(\nu)}|_{S_1} = 0$, $t^{(\nu)}(u_0)|_{S_1} \geqslant 0$ or $u_0^{(\nu)}|_{S_1} > 0$, $t^{(\nu)}(u_0)|_{S_1} = 0$, and the condition $t^{(\nu)}(u_0)|_{S_2} = f(x)$ on S_2.

We later use the method /6,7/ that results in a unilateral variational problem for certain boundary functionals.

Let a vector $u^*(x)$, $x \in \bar{G}$ satisfying the conditions

$$Au^* = K \text{ in } G, \quad u^*|_{S_1} = 0, \quad t^{(\nu)}(u^*)|_{S_2} = f \tag{1.2}$$

exist. Then /7/

$$2\int_G W(u^*, w)\,dG = \int_G Kw\,dG + \int_{S_2} fw\,ds + \int_{S_1} t^{(\nu)}(u^*)\,w\,ds, \quad \forall w \in W_2^1(G) \tag{1.3}$$

To obtain this relationship it is sufficient to multiply both sides of the equality

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$Au^* = K$ by $w \in W_2^1(G)$ to integrate over G and to apply the Betti formula /8/. Taking $w = v - u_0 \in W_2^1(G)$ in (1.3) and subtracting the identity (1.3) from (1.1), we obtain

$$2\int_G W(u_0 - u^*, v - u_0)\, dG \geqslant -\int_{S_1} t^{(\nu)}(u^*)(v - u_0)\, ds, \quad \forall v \in V$$

Let us set $u_0 - u^* = \varphi_0 \in W_2^1(G)$ here (since $u^*|_{S_1} = 0$, then $\varphi_0|_{S_1} = u_0|_{S_1} \geqslant 0$, i.e. $\varphi_0 \in V$) then we arrive at the inequality

$$2\int_G W(\varphi_0, v - u_0)\, dG \geqslant -\int_{S_1} t^{(\nu)}(u^*)(v - u_0)\, ds, \quad \forall v \in V \tag{1.4}$$

The interpretation of this variational inequality (similarly to /2/ or /3/) results in the following boundary value problem for the vector φ_0 with unilateral constraints in S_1:

$$A\varphi_0 = 0 \text{ in } G, \quad \varphi_0|_{S_1} \geqslant 0, \quad t^{(\nu)}(\varphi_0) + t^{(\nu)}(u^*)|_{S_1} \geqslant 0 \quad \varphi_0 [t^{(\nu)}(\varphi_0) + t^{(\nu)}(u^*)]_{S_1} = 0, \quad t^{(\nu)}(\varphi_0)|_{S_2} = 0 \tag{1.5}$$

For the functions φ satisfying the equation $A\varphi = 0$ in G, $t^{(\nu)}(\varphi)|_{S_2} = 0$ by virtue of the Betti formula /8/, the following equality holds

$$2\int_G W(\varphi, w)\, dG = \int_{S_1} t^{(\nu)}(\varphi)\, w\, ds \quad \forall w \in W_2^1(G) \tag{1.6}$$

Taking this into account, for $w = v - u_0$ and $u_0|_{S_1} = \varphi_0|_{S_1} \in V$ we obtain from (1.4) a unilateral variational problem for the boundary functionals

$$\int_{S_1} t^{(\nu)}(\varphi_0)(v - \varphi_0)\, ds \geqslant -\int_{S_1} t^{(\nu)}(u^*)(v - \varphi_0)\, ds, \quad \forall v \in V \tag{1.7}$$

Following /1/, we can show that the problem for the variational inequality (1.7) is equivalent to the problem of minimizing the functional

$$F(\varphi) = \frac{1}{2}\int_{S_1} t^{(\nu)}(\varphi)\, \varphi\, ds + \int_{S_1} t^{(\nu)}(u^*)\, \varphi\, ds, \quad \varphi \in V$$

in the set V, where $t^{(\nu)}(u^*)$ is a known element. The inequality (1.7) here corresponds to the condition $F'(\varphi_0)(v - \varphi_0) \geqslant 0$, $\forall v \in V$. Assuming that the Korn inequality /1/ is satisfied for the problem (1.5), which means that the following condition holds

$$\int_G \operatorname{rot} \varphi\, dG = 0$$

we investigate the solvability of the problem of minimizing the functional $F(\varphi)$ on V.

The symmetry of the boundary bilinear form $\int w t^{(\nu)}(\varphi) ds$ and the positivity of the corresponding quadratic form $\int \varphi t^{(\nu)}(\varphi) ds$ (integration over S_1) follows from (1.6) if the vector-function $\varphi \in V$ is subject to the condition

$$\int_G \varphi\, dG = 0 \tag{1.8}$$

Then the functions $\varphi|_{S_1}$ belong to the subspace $W_2^{*1/2}(S_1) \subset W_2^{1/2}(S_1)$ constructed in /6/ with the scalar product

$$[w, \varphi]_{1/2, S_1} = \int_{S_1} w t^{(\nu)}(\varphi)\, ds_1$$

($W_2^{1/2}(S_1)$ is the Sobolev–Slobodetskii space). The set V_{S_1} (the set of traces of the vector-function $\varphi \in V$ on S_1) is evidently closed in $W_2^{*1/2}(S_1)$ /6/. The operator T generated by the continuous bilinear form $\langle w, t^{(\nu)}(\varphi)\rangle$ as the reciprocity ratio on $W_2^{*1/2}(S_1) \times W_2^{-1/2}(S_1)$ is, by the Riesz theorem, the isometry $W_2^{*1/2}(S_1)$ on $W_2^{-1/2}(S_1)$ defined by the relationship /6/

$$[w, \varphi]_{1/2, S_1} = (w, T\varphi)_{0, S_1} = \langle w, t^{(\nu)}(\varphi)\rangle, \quad \forall w, \varphi \in W_2^{*1/2}(S_1) \tag{1.9}$$

The operator T^{-1} is the mapping of $W_2^{-1/2}(S_1)$ on V that satisfies the Lipschitz condition (see /1/, Theorem 2.5). Therefore, the solution $\varphi_0 \in V_{S_1} \subset W_2^{*1/2}(S_1)$ that minimizes the problem for the functional $F(\varphi)$ exists for any given vector $t^{(\nu)}(u^*) \in W_2^{-1/2}(S_1)$ satisfying the conditions

$$\int_{S_1} t^{(\nu)}(u^*)\, ds = 0, \quad \int_{S_1} t^{(\nu)}(u^*) \times r\, ds = 0$$

where r is the radius-vector of the point $x \in S_1$, and this solution is unique /2,3/.

Remark. It follows from (1.9) that the scalar product $(,)_{0,S_1}$ of elements from $W_2^{*1/2}(S_1) \times W_2^{-1/2}(S_1)$ in $L_2(S_1)$ is identified with the reciprocity relation $\langle,\rangle$ in $W_2^{*1/2}(S_1) \times W_2^{-1/2}(S_1)$.

The following equality is needed in the sequel /5,6/:

$$\langle \varphi, t^{(\nu)}(\varphi)\rangle \equiv (\varphi, T\varphi)_{0,S_1} = (t^{(\nu)}(\varphi), T^{-1}t^{(\nu)}(\varphi))_{0,S_1}, \quad \forall\varphi \in W_2^{*1/2}(S_1) \tag{1.10}$$

and it follows from (1.9) and the fact that the operator T^{-1} is also an isometry $W_2^{-1/2}(S_1)$ in $W_2^{*1/2}(S_1)$, and therefore, the equality holds /6/

$$\| T\varphi \|_{-1/2, S_1} = | T^{-1}t^{(\nu)}(\varphi) |_{1/2, S_1}, \quad \forall\varphi \in W_2^{*1/2}(S_1)$$

Let us discuss the question of the existence and uniqueness of the solution of the Signorini problem from the aspect of the approach elucidated for its study, which uses the introduction of an auxiliary mixed boundary value problem (1.2) with zero boundary condition for the vector u^* in the zone of possible contact. As is known /1/, the solution of the generalized Signorini problem exists to the accuracy of a rigid displacement ρ such that $\Psi(\rho) = 0$ and $u_0 + \rho \in V$ under the conditions

$$\Psi(\rho) \equiv 2\int_G K\rho\, dG + 2\int_{S_2} f\rho\, ds \leqslant 0 \quad \forall\rho \in R' = R \cap V$$

where R the space of rigid displacements, is a kernel of quadratic form

$$2\int_G W(v)\, dG$$

and the equality sign holds if and only if $\rho \in R^* = \{\rho \in R' \mid \rho \in R \Rightarrow -\rho \in R$ is a subset of bilateral vectors from R'.

The approach elucidated above also permits the proof of the existence of a solution of the problem (1.1): the solution u^* of the problem (1.2) exists and is unique, the solution φ_0 of the unilateral problem (1.7) also exists uniquely; then there exists a solution $u_0 = u^* + \varphi_0$ of the problem (1.1) and any other solution u_0' of this problem is $u_0' = u_0 + \rho$, where ρ is a rigid displacement such that $\Psi(\rho) = 0$, $u_0 + \rho \in V$.

In the case of the contact problem for a system of linearly deformable bodies, a sufficiently complete analysis of the solvability conditions has been carried out earlier /9/ for the corresponding unilateral variational problem on the basis of the Lions–Stampacci theorem.

2. In making the transition to reciprocity formulations of variational problems with unilateral constraints, different approaches can be utilized that reduce the problem constraints, in particular, the Lagrange multiplier method /3,4,10,11/, the method of conjugate functions /2–4,12,13/. The latter is used below.

Let us introduce the function $\varphi \to \psi(\varphi)$ defined as follows in the space $W_2^{*1/2}(S_1)$

$$\psi(\varphi) = \begin{cases} \langle \varphi, t^{(\nu)}(u^*)\rangle, & \varphi \in V \\ +\infty, & \varphi \,\overline{\in}\, V \end{cases}$$

(see /3/ for the properties of the function $\psi(\varphi)$). Then the problem of finding $\inf F(\varphi)$ in $\varphi \in V$ reduces to the problem of finding

$$\inf_{\varphi \in W_2^{*1/2}(S_1)} \left[\frac{1}{2}(\varphi, T\varphi)_{0,S_1} + \psi(\varphi)\right] \tag{2.1}$$

(here the relationship (1.9) is used).

Let us define the function ψ^* conjugate to ψ with respect to the reciprocity relation $\langle \varphi, -t^{(\nu)}(\varphi)\rangle$ on $W_2^{*1/2}(S_1) \times W_2^{-1/2}(S_1)$ as follows:

$$\psi^*(-t^{(\nu)}(\varphi)) = \sup_{\varphi \in W_2^{*1/2}(S_1)} [\langle \varphi, -t^{(\nu)}(\varphi)\rangle - \psi(\varphi)]$$

(in some sources /3,4,12/, the function ψ^* is called the Young–Fenchel–Moreau transform of the function ψ, see /3,4/) for the properties of ψ^*

The following relationship holds /4,14/:

$$\psi(\varphi) + \psi^*(-t^{(\nu)}(\varphi)) - \langle \varphi, -t^{(\nu)}(\varphi)\rangle \geqslant 0, \quad \forall\varphi \in W_2^{*1/2}(S_1), \quad t^{(\nu)}(\varphi) \in W_2^{-1/2}(S_1) \tag{2.2}$$

The conjugate to the functional

$$f(\varphi) = \frac{1}{2}(\varphi, T\varphi)_{0,S_1}, \quad \varphi \in W_2^{*1/2}(S_1)$$

is the functional /5/

$$f^*(t^{(\nu)}(\varphi)) = \frac{1}{2}(t^{(\nu)}(\varphi), T^{-1}t^{(\nu)}(\varphi))_{0,S_1} = f^*(-t^{(\nu)}(\varphi)), \quad t^{(\nu)}(\varphi) \in W_2^{-1/2}(S_1)$$

and the following relationships hold /14/:

$$\begin{gathered} f(\varphi) + f^*(t^{(\nu)}(\varphi)) - \langle \varphi, t^{(\nu)}(\varphi)\rangle \geqslant 0 \\ f(\varphi_0) + f^*(T\varphi_0) - \langle \varphi_0, T\varphi_0\rangle = 0 \\ \forall \varphi \in W_2^{*1/2}(S_1), \quad t^{(\nu)}(\varphi) \in W_2^{-1/2}(S_1) \end{gathered} \tag{2.3}$$

(see the relationship (2.3) in /5/, also).

We shall henceforth confine ourselves to the scheme in /5/: we prove the reciprocity relationship and we investigate the possibility of utilizing the estimates obtained in /5/ for the approximate solutions of the problems (1.1) and (2.1).

Theorem 1. The functionals

$$\begin{gathered} F(\varphi) = f(\varphi) + \psi(\varphi), \quad \varphi \in W_2^{*1/2}(S_1) \\ \Phi(-t^{(\nu)}(\varphi)) = -[f^*(-t^{(\nu)}(\varphi)) + \psi^*(-t^{(\nu)}(\varphi))], \quad t^{(\nu)}(\varphi) \in W_2^{-1/2}(S_1) \end{gathered}$$

form a reciprocal pair, i.e., the following relationship is satisfied

$$F(\varphi_0) = \inf_{\varphi \in W_2^{*1/2}(S_1)} F(\varphi) = \sup_{t^{(\nu)}(\varphi) \in W_2^{-1/2}(S_1)} \Phi(-t^{(\nu)}(\varphi)) = \Phi(-t^{(\nu)}(\varphi_0)), \quad \varphi_0 \in V$$

Following /5/, and taking account of (1.10) and (2.2), we obtain

$$\begin{gathered} F(\varphi) - \Phi(-t^{(\nu)}(\varphi)) = \frac{1}{2}\langle \varphi, t^{(\nu)}(\varphi)\rangle + \psi(\varphi) + \\ \frac{1}{2}\langle \varphi, t^{(\nu)}(\varphi)\rangle + \psi^*(-t^{(\nu)}(\varphi)) = \\ \psi(\varphi) + \psi^*(-t^{(\nu)}(\varphi)) + \langle \varphi, t^{(\nu)}(\varphi)\rangle \geqslant 0 \\ \forall \varphi \in W_2^{*1/2}(S_1), \quad t^{(\nu)}(\varphi) \in W_2^{-1/2}(S_1) \end{gathered} \tag{2.4}$$

Let φ_0 be an element from $V\,(\varphi_0|_{S_1} \geqslant 0)$ on which $\inf F(\varphi)$ and $\sup \Phi(-t^{(\nu)}(\varphi))$ are achieved. Then taking account of (1.9), (2.3), and the definition of the functions $\psi(\varphi), \psi^*(-t^{(\nu)}(\varphi))$, we obtain

$$\begin{gathered} F(\varphi_0) - \Phi(-t^{(\nu)}(\varphi_0)) = f(\varphi_0) + f^*(T\varphi_0) + \psi(\varphi_0) + \\ \psi^*(-t^{(\nu)}(\varphi_0)) = \langle \varphi_0, T\varphi_0\rangle + \langle \varphi_0, t^{(\nu)}(u^*)\rangle + \\ \sup_{\varphi \in W_2^{*1/2}(S_1)} [\langle \varphi_0, -t^{(\nu)}(\varphi_0) - t^{(\nu)}(u^*)\rangle] = \\ \langle \varphi_0, T\varphi_0\rangle + \langle \varphi_0, t^{(\nu)}(u^*)\rangle - \langle \varphi_0, t^{(\nu)}(\varphi_0)\rangle - \langle \varphi_0, t^{(\nu)}(u^*)\rangle = 0 \end{gathered}$$

Hence, and from (2.4) the proof of the theorem follows.

Therefore, it is also proved that the mapping $\varphi \to t^{(\nu)}(\varphi)$ sets up a connection between the solutions of the initial and reciprocal problems.

Let $\varphi_n \in V_n$ be an approximate solution of the problem of the minimum of the functional $F(\varphi)$, where V_n is a finite dimensional approximation of V constructed by some method /3,10/.

Theorem 2. Let $\varphi_n \in V_n$ be an approximation for the element $\varphi \in V$ that realizes the minimum of $F(\varphi)$ in the set V such that $\varphi_n \to \varphi_0$ as $n \to \infty$ in the metric $W_2^1(G)$ (this means in the metrics $W_2^{1/2}(S_1)$, $W_2^{-1/2}(S_1)$ also), then the error estimates hold /5/

$$\begin{gathered} \| u_0 - u_n \|_{1,G} = \| \varphi_0 - \varphi_n \|_{1,G} \leqslant \Delta(\varphi_n) \\ \| u_0 - u_n \|_{1/2,S_1} = \| \varphi_0 - \varphi_n \|_{1/2,S_1} \leqslant c_1 \Delta(\varphi_n), \quad c_1 > 0 \\ \| t^{(\nu)}(u_0) - t^{(\nu)}(u_n) \|_{-1/2,S_1} = \| t^{(\nu)}(\varphi_0) - t^{(\nu)}(\varphi_n) \|_{-1/2,S_1} \leqslant c_2 \Delta(\varphi_n), \quad c_2 > 0 \\ \Delta(\varphi_n) = \left\{\frac{1}{c}[F(\varphi_n) - \Phi(-t^{(\nu)}(\varphi_n))]\right\}^{1/2} \end{gathered}$$

Here $c > 0$ is the constant from the inequalities

$$2\int_G W(\varphi)\,dG \geqslant c\|\varphi\|_{1,G}^2, \quad \|\cdot\|_{1,G} = \|\cdot\|_{W_2^1(G)} \tag{2.5}$$

which holds for functions $\varphi \in W_2^1(G)$ satisfying condition (1.8).

In its main features the proof of the theorem duplicates the proof of the corresponding theorem from /5/.

The element φ_0 minimizes the functional

$$F(\varphi) = \int_G W(\varphi)\,dG + \int_{S_1} \varphi t^{(\nu)}(u^*)\,ds, \quad \varphi \in V$$

(here (1.6) is utilized), which hence satisfies the variational inequality (1.7). Let us form the difference of the functionals

$$F(\varphi_n) - F(\varphi_0)$$

The inequality

$$2\int_G W(\varphi_0, \varphi_n - \varphi_0)\,dG = \int_{S_1} t^{(\nu)}(\varphi_0)(\varphi_n - \varphi_0)\,ds \geqslant \int_{S_1} t^{(\nu)}(u^*)(\varphi_0 - \varphi_n)\,ds, \qquad \forall \varphi_n \in V_n$$

follows from (1.7) and (1.6). Using it, we obtain

$$F(\varphi_n) - F(\varphi_0) \geqslant \int_G W(\varphi_n)\,dG - \int_G W(\varphi_0)\,dG -$$
$$2\int_G W(\varphi_0, \varphi_n - \varphi_0)\,dG = \int_G W(\varphi_n)\,dG - \int_G W(\varphi_0)\,dG -$$
$$2\int_G W(\varphi_0, \varphi_n)\,dG + 2\int_G W(\varphi_0)\,dG = \int_G W(\varphi_n - \varphi_0)\,dG$$

Hence, by virtue of (2.5), the inequality

$$F(\varphi_n) - F(\varphi_0) \geqslant c' \|\varphi_n - \varphi_0\|^2_{1,G}, \quad c' = {}^1/_2 c$$

follows.

Then by using the inequality $F(\varphi_0) > \Phi(-t^{(\nu)}(\varphi_n))$ resulting from the reciprocity relation of Theorem 1, we obtain the first estimate of the theorem. From the inequalities of the theorem of traces on S for functions from $W_2^1(G)$, the second and third estimates of the theorem follow /5/.

Therefore, if a vector $u^*(x)$ is constructed that satisfies the identity (1.3) (such a vector can be the solution of the mixed problem of elasticity theory (1.2) constructed by the generalized Trefftz method, say /7/), and approximate solutions $\varphi_n \in V_n$ are constructed for the variational problem (1.7), then the approximate solutions of the unilateral Signorini problem (1.1) are determined from the rule $u_n = u^* + \varphi_n$.

Analogously to /5/, the problem of minimizing $F(\varphi)$ on V can also be formulated as a problem of finding the projection of the element $t^{(\nu)}(u^*)$.

It follows from (1.10) that the metric in elements from $W_2^{-1/2}(S_1)$ can also be introduced as follows (see /5/):

$$|t^{(\nu)}(\varphi)|_{-1/2, S_1} = \{(t^{(\nu)}(\varphi), T^{-1}t^{(\nu)}(\varphi))_{0, S_1}\}^{1/2}$$

Then the functional $F(\varphi)$ can be written in the form (see (1.10)

$$F(\varphi) = \frac{1}{2}|t^{(\nu)}(\varphi)|^2_{-1/2, S_1} + [t^{(\nu)}(u^*), t^{(\nu)}(\varphi)]_{-1/2, S_1} = \frac{1}{2}|t^{(\nu)}(\varphi) + t^{(\nu)}(u^*)|^2_{-1/2, S_1} - \frac{1}{2}|t^{(\nu)}(u^*)|^2_{-1/2, S_1}$$

If the set $V^* \subset W_2^{-1/2}(S_1)$ conjugate to V /3/ is introduced, then the problem of finding $\inf F(\varphi)$ in V is a problem of finding the projection of the element $t^{(\nu)}(u^*) \in W_2^{-1/2}(S_1)$ in the set V^* /5/, and

$$\inf_{t^{(\nu)}(\varphi) \in V^*} |t^{(\nu)}(\varphi) + t^{(\nu)}(u^*)|_{-1/2, S_1} = |t^{(\nu)}(\varphi_0) + t^{(\nu)}(u^*)|_{-1/2, S_1}$$

3. Let us present certain reasoning about the practical utilization of error estimates obtained in Theorem 2 for the approximate solution u_n of the Singorini problem $(u_0 - u_n = \varphi_0 - \varphi_n)$.

If $\varphi_n \in V_n$ is an approximate solution of the problem of minimizing the functional

$$F(\varphi) = \frac{1}{2}\int_{S_1} \varphi t^{(\nu)}(\varphi)\,ds + \int_{S_1} \varphi t^{(\nu)}(u^*)\,ds$$

on V, then according to what has been proved the approximate solution of the reciprocal problem for the functional $\Phi(-t^{(\nu)}(\varphi))$ is $-t^{(\nu)}(\varphi_n)$ where

$$t^{(\nu)}(\cdot) = \sum_{i,k,l,m=1}^{3} c_{iklm}(x)\,\varepsilon_{lm}(\cdot)\cos(\nu, x_i)\,x_k^{(0)}$$

$x \in S_1$ is a vector-operator of the boundary stresses /6/.

The difficulties in the practical realization of the reciprocity principle utilized above are /3/ the construction of expressions for the function ψ^* conjugate to ψ. In order to write down the expression for the function ψ^* explicitly in the problem under investigation, additional constraints must be imposed on $\varphi \in V$, namely, if $\varphi|_{1/2, S_1} \leqslant 1$, then

$$\psi^*(-t^{(\nu)}(\varphi)) = |-t^{(\nu)}(\varphi) - t^{(\nu)}(u^*)|_{-1/2, S_1}$$

At the same time utilization of the error estimates obtained above for a known approximate solution φ_n does not require explicit representatation of the functional $\Phi(-t^{(\nu)}(\varphi_n))$ since the following estimate holds for the difference in the functionals $F(\varphi_n) - \Phi(-t^{(\nu)}(\varphi_n))$.

$$F(\varphi_n) - \Phi(-t^{(\nu)}(\varphi_n)) = \frac{1}{2}\langle\varphi_n, t^{(\nu)}(\varphi_n)\rangle + \psi(\varphi_n) +$$
$$\frac{1}{2}\langle\varphi_n, t^{(\nu)}(\varphi_n)\rangle + \psi^*(-t^{(\nu)}(\varphi_n)) = \langle\varphi_n, t^{(\nu)}(\varphi_n) + t^{(\nu)}(u^*)\rangle +$$
$$\sup_{\varphi_n \in V_n} [-\langle\varphi_n, t^{(\nu)}(\varphi_n) + t^{(\nu)}(u^*)\rangle] \leqslant \langle\varphi_n, t^{(\nu)}(\varphi_n) + t^{(\nu)}(u^*)\rangle +$$
$$\langle\varphi_n, t^{(\nu)}(\varphi_n) + t^{(\nu)}(u^*)\rangle = 2I(\varphi_n)$$
$$I(\varphi_n) = \int_{S_1} \varphi_n [t^{(\nu)}(\varphi_n) + t^{(\nu)}(u^*)]\, ds$$

We here used the relationship

$$\sup_{\varphi_n \in V_n} [-\langle\varphi_n, t^{(\nu)}(\varphi_n) + t^{(\nu)}(u^*)\rangle] = -\inf_{\varphi_n \in V_n} \langle\varphi_n, t^{(\nu)}(\varphi_n) + t^{(\nu)}(u^*)\rangle \leqslant \langle\varphi_n, t^{(\nu)}(\varphi_n) + t^{(\nu)}(u^*)\rangle$$

since $I(\varphi_n) \geqslant 0$ by virtue of (1.5). Also by virtue of (1.5), the following condition

$$\lim_{n\to\infty} I(\varphi_n) = I(\varphi_0) = 0$$

is satisfied. Therefore, the right side of the error estimates obtained $\{c^{-1}2I(\varphi)\}^{1/2}$ is convenient for calculations (see /5/ also).

The efficiency of the approach proposed for the numerical solution of contact problems of elasticity theory is firstly in the reduction of the dimensionality of the problem being solved because of the reduction of a three-dimensional problem to the solution of equations defined only on the domain boundary (in the zone of possible contact), secondly, in the explicit relationship between the solutions of the initial and reciprocal problems, which facilitates utilization of error estimates of the approximate solution. At the same time, the utilization of such a widespread method as the finite-element method for the construction of the solution is difficult because the coordinate functions in the problem (1.5) should satisfy the elasticity theory equation in the ordinary sense, i.e., be twice continuously differentiable.

The spaces of the traces $W_2^{*1/2}(S_1) \times W_2^{-1/2}(S_1)$ figure in the variational principle constructed above (Theorem 1). Utilization of the spaces $W_2^{-1/2}(S) \times W_2^{1/2}(S)$ is ordinarily associated with difficulties in the practical calculation of their norms since norms with fractional index are singular repeated integrals on the domain boundary, and calculation of the norm in conjugate space (negative) is related to the solution of the auxiliary maximization problem. The abovementioned difficulties are not present in the variational principle constructed since the norm in the space $W_2^{*1/2}(S_1)$ is defined in conformity with (1.9) and is equivalent /6/ to the norm in $W_2^{1/2}(S_1)$. Evaluation of the norm in the reciprocal space $W_2^{-1/2}(S_1)$ is not required since if φ_n is the approximate solution found for the initial problem, then $-t^{(\nu)}(\varphi_n)$ is the solution of the reciprocal problem, and therefore, there is no need to solve the reciprocal problem related to the calculation of the norm in $W_2^{-1/2}(S)$.

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